Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(99)00043-X
J. Appl. Maths Mechs, Vol. 63, No. 2, pp. 327–332, 1999

0021-8928/99/\$—see front matter

A VARIATIONAL METHOD OF DERIVING THE EQUATIONS OF THE NON-LINEAR MECHANICS OF LIQUID CRYSTALS†

V. B. LISIN and A. I. POTAPOV

Nizhnii Novgorod

(*Received* 20 *April* 1998)

The non-linear equations of the dynamics of liquid crystals [1], derived previously by the Poisson brackets method, are derived from the Hamilton–Ostrogradskii variational principle. The variational problem of an unconditional extremum of the action functional in Lagrange variables is investigated. The difference between the volume densities of the kinetic and free energy of the liquid crystal is used as the Lagrangian. It is shown that the variational equations obtained are equivalent to the differential laws of conservation of momentum and the kinetic moment of the liquid crystal in Euler variables, while the Ericksen stress tensor and the molecular field are defined in terms of the derivatives of the free energy. © 1999 Elsevier Science Ltd. All rights reserved.

A liquid crystal is a fluid medium in which, unlike classical liquids, there is an additional hydrodynamic variable—the director field, which describes the orientational motions of stretched particles [2]. The equations of the non-linear dynamics of liquid crystals were derived in [1] by the Poisson brackets method, which are well known in problems of superfluidity. However, it is fairly complex and length in the theory of liquid crystals compared with the variational method employed below, the use of which is well known in the case of classical liquids [3, 4].

1. THE KINEMATIC AND ENERGY CHARACTERISTICS

The dynamic state of a nematic liquid crystal as a continuous medium is described by a specified velocity field $\mathbf{v}(\mathbf{x}, t)$, a density field $\rho(\mathbf{x}, t)$ and a pressure field $\rho(\mathbf{x}, t)$, and also by the field of the directions of the particles of the medium $\mathbf{n}(\mathbf{x}, t)$ (the director field) [5, 6]. For a variational derivation of the equations of the dynamics of a nematic liquid crystal it is necessary to obtain an expression for the Lagrangian of a physically infinitesimal element of the medium. In continuum mechanics, the Lagrangian has an energy meaning and is chosen to be equal to the difference between the kinetic energy density and the internal or free energy density, depending on the nature of the processes being investigated.

In the case considered here, the volume density of the kinetic energy is

$$K = \rho \mathbf{v}^2/2 + \rho J\omega^2/2, \quad \omega_s = (\mathbf{n} \times d\mathbf{n}/dt)_s = \epsilon_{ski}\, n_s \dot{n}_k \tag{1.1}$$

The first term on the right-hand side of the first relation of (1.1) describes the kinetic energy, related to the translational motion of the centre of mass of a physically infinitesimal volume, while the second term describes the energy related to the rotation of the molecules about the centre of mass, J is the geometrical moment of inertia, ω is the angular velocity of rotation of the director and ε_{ski} are the components of the Levi–Civita pseudotensor. Here we have also taken into account the fact that the vector $\mathbf{n}$ has unit length, and all the changes of the director are related to its rotation in space.

In the simplest case for a nematic liquid crystal the free energy has the form [2]

$$\rho F = \rho F_0(\rho) + \frac{1}{2}K_1(\operatorname{div}\mathbf{n})^2 + \frac{1}{2}K_2(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})^2 + \frac{1}{2}K_3(\mathbf{n}\times\operatorname{rot}\mathbf{n})^2 - \frac{1}{8\pi}\Delta\varepsilon(\mathbf{n}\cdot\mathbf{E})^2 - \frac{1}{2}\Delta\chi(\mathbf{n}\cdot\mathbf{H})^2 \tag{1.2}$$

Here K_i are Frank constants, and $\Delta\varepsilon$ and $\Delta\chi$ are the dielectric and diamagnetic anisotropy. In the mechanics of liquid crystals, the free energy plays a role similar to that of the elastic deformation energy of a solid and gives it some similarities with the theory of elasticity. The first term on the right-hand side of (1.2) describes the hydrodynamic part of the free energy, in terms of which the pressure in the medium $p = \rho^2(\partial F/\partial\rho)$ is expressed, the second and third terms are related to the director field gradient, while the last two describe the interaction between the electric field E and magnetic field H and the director field n. They are responsible for the change in the orientation of the director due to the action of the electric and magnetic fields. The three combinations with coefficients K_i which occur in (1.2), are independent of one another; each of them can be non-zero when the other

†*Prikl. Mat. Mekh.* Vol. 63, No. 2, pp. 340–346, 1999.

two are equal to zero. Deformations in which only one of the quantities div $\mathbf{n}$, $(\mathbf{n}\cdot\operatorname{rot}\mathbf{n})$ or $\mathbf{n}\times\operatorname{rot}\mathbf{n}$ are non-zero are called transverse bending, torsion or longitudinal bending respectively. Corresponding to this, K_1 is sometimes called the modulus of elasticity of transverse bending, K_2 is called the torsion modulus, while K_3 is called the longitudinal bending modulus. The volume density of the Lagrange function in Euler variables is

$$L(\rho, \mathbf{v}, n_k, n_{k,i}, \omega) = \rho\{\mathbf{v}^2/2 + J\omega^2/2 - F(\rho, n_k, n_{k,i}, \mathbf{E}, \mathbf{H})\} \tag{1.3}$$

Here and henceforth the subscript after the comma denotes a derivative with respect to the corresponding coordinate (for example, $n_{k,i} = \partial n_k/\partial x_i$).

2. THE VARIATIONAL EQUATIONS

There are two equivalent approaches to the variational derivation of the equations of continuum dynamics [3–5]: either one can consider the variational problem on a conditional extremum for the action functional, written in Euler variables, or one can investigate the variational problem on an unconditional extremum of the action functional, written in Lagrange variables.

In the first approach, one needs to know additional relations imposed on the field variables. These relations, in the case of media with an internal structure, such as liquid crystals, are not known in advance.

In the second approach, which is used in the present paper, all the field variables occurring in (1.3) must be expressed in terms of independent Lagrange variables, which are the coordinates of the centre of mass of the particles $\xi_\alpha(x_i, t)$ and their orientations $n_\alpha(x_i, t)$, $\alpha = 1, 2, 3$. Here and henceforth Latin subscripts denote Euler coordinates, while Greek subscripts denote Lagrange coordinates.

The Euler velocity field can be expressed in terms of the Lagrange coordinates ξ_α and their derivatives from the condition for the conservation of the Lagrange coordinates of a particle $d\xi_\alpha/dt = \partial\xi_\alpha/\partial t + \mathrm{v}_j\xi_\alpha/\partial x_j = 0$. Hence, we obtain that

$$\mathrm{v}_j = -\frac{\partial x_j}{\partial \xi_\alpha}\frac{\partial \xi_\alpha}{\partial t} \tag{2.1}$$

From the equation of continuity of the medium in Lagrange variables we obtain

$$\rho[x_i(\xi_\alpha), t] = \rho_0(\xi_\alpha)\det\|\xi_{\alpha,i}\|/\det\|\xi^0_{\alpha,i}\| \tag{2.2}$$

where ρ is the density while ξ^0_α are the Lagrange coordinates of the medium at the initial instant of time. Hence, expressing the field quantities ρ, $\mathbf{v}$, $\boldsymbol{\omega}$ using relations (2.1) and (2.2), we obtain the Lagrangian (1.3) as a function of the independent variables ξ_α, n_k and their derivatives

$$L = L(\xi_{\alpha,j}, \xi_{\alpha,t}, n_k, \dot n_k, n_{k,j}) \tag{2.3}$$

$$(\xi_{\alpha,t} = \partial\xi_\alpha/\partial t, \quad \dot n_k = dn_k/dt = \partial n_k/\partial t + \mathrm{v}_j\partial n_k/\partial x_j)$$

The subscript t after the comma denotes the partial derivative with respect to time of the function for fixed Euler coordinates x_i, while a dot denotes a derivative with respect to time for fixed Lagrangian coordinates ξ_α.

It follows from the Hamilton–Ostrogradskii variational principle in Lagrange variables that the motion of a continuous medium corresponds to an unconditional extremum of the action functional

$$I[\xi_\alpha, n_k] = \int_{t_0}^{t_1}\int_V L(\xi_{\alpha,i}, \xi_{\alpha,t}, n_k, \dot n_k, n_{k,j})dVdt \tag{2.4}$$

The integration is carried out over the time-varying volume $V = V(t)$, occupied by the same particles of the medium which cannot cross its boundary.

By varying functional (2.4) with respect to the independent variables ξ_α and n_k for constant Euler coordinates x_j (see the Appendix) and using Green's formula for separating the variation with respect to the volume $V(t)$ and its surface $\Sigma = \Sigma(t)$, we obtain

$$\delta_x I[\xi_\alpha, n_k] = \int_{t_0}^{t_1}\int_V \left\{\left[-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \xi_{\alpha,j}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \xi_{\alpha,t}}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot n_k}n_{k,j}\right)x_{j,\alpha} + \frac{\partial}{\partial x_m}\left(\frac{\partial L}{\partial \dot n_k}\mathrm{v}_m n_{k,j}\right)x_{j,\alpha} +\right.\right.$$

$$\left.\left. + \frac{\partial L}{\partial \dot n_k}n_{k,m}\mathrm{v}_{m,j}x_{j,\alpha}\right]\delta_x\xi_\alpha + \left[\frac{\partial L}{\partial n_k} - \frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial n_{k,j}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot n_k}\right) - \frac{\partial L}{\partial \dot n_k}\mathrm{v}_{m,m}\right]\delta_x n_k\right\}dVdt +$$

$$+\int_{t_0}^{t_1}\int_{\Sigma(t)}\left\{\left[\left(\frac{\partial L}{\partial \xi_{\alpha,j}}-\frac{\partial L}{\partial \dot n_k}\mathrm{v}_j n_{k,i}x_{i,\alpha}\right)\mathrm{v}_j-\left(\frac{\partial L}{\partial \xi_{\alpha,t}}-\frac{\partial L}{\partial \dot n_k}x_{i,\alpha}n_{k,i}\right)C_\Sigma\right]\delta_x\xi_a+\right.$$
$$\left.+\left[\left(\frac{\partial L}{\partial n_{k,j}}+\frac{\partial L}{\partial \dot n_k}\right)\mathrm{v}_j-\frac{\partial L}{\partial \dot n_k}C_\Sigma\right]\delta_x n_k\right\}d\Sigma dt+$$
$$+\left\{\int_V\left[\left(\frac{\partial L}{\partial \xi_{\alpha,t}}-\frac{\partial L}{\partial \dot n_k}x_{i,\alpha}n_{k,i}\right)\delta_x\xi_\alpha+\left(\frac{\partial L}{\partial \dot n_k}\right)\delta_x n_k\right]dV\right\}\Bigg|_{t_0}^{t_1}=0 \tag{2.5}$$

Here v_j are the components of the normal to the surface Σ while C_Σ is the normal velocity of points of the surface $\Sigma(t)$.

Assuming that the variations $\delta_x\xi_\alpha$ and $\delta_x n_k$ vanish on the surface of the volume, and that the constants of the system at the beginning $t = t_0$ and at the end $t = t_1$ of the motion are known, we obtain that the second and third integrals in (2.5) are zero, while the first integral reduces to the variational equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \xi_{\alpha,t}}\right)+\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial \xi_{\alpha,j}}\right)=\left(\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot n_k}n_{k,j}\right)+\frac{\partial}{\partial x_m}\left(\frac{\partial L}{\partial \dot n_k}\mathrm{v}_m n_{k,j}\right)+\frac{\partial L}{\partial \dot n_k}\mathrm{v}_{m,j}n_{k,m}\right)\frac{\partial x_j}{\partial \xi_\alpha} \tag{2.6}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot n_k}\right)+\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial n_{k,j}}\right)-\frac{\partial L}{\partial n_k}=-\frac{\partial L}{\partial \dot n_k}\mathrm{v}_{m,m} \tag{2.7}$$

The first of these describes translational motions in the liquid crystal while the second describes the dynamics of the director.

From the variation of the action integral (2.5) one can also obtain the boundary conditions on the surface of the volume occupied by the liquid crystal. To do this we need to assume that the variations $\delta_x\xi_\alpha$ and $\delta_x n_k$ are not zero on the surface Σ, and Eqs (2.6) and (2.7) hold and, moreover, the conditions of impermeability of the boundary Σ: $C_\Sigma = 0$ are satisfied. Then, from the condition for the first and third integrals in (2.5) to vanish we obtain the natural boundary conditions

$$\left.\frac{\partial L}{\partial \xi_{\alpha,j}}\mathrm{v}_j-\frac{\partial L}{\partial \xi_{\alpha,t}}G_\Sigma\right|_\Sigma=0,\quad \left.\frac{\partial L}{\partial n_{k,j}}\mathrm{v}_j\right|_\Sigma=0 \tag{2.8}$$

3. THE EQUIVALENCE OF THE VARIATIONAL EQUATIONS TO THE LOCAL CONSERVATION LAWS

We will return, in Eq. (2.6), to the hydrodynamic variables ρ, v_j, p using the conversion formulae (2.1) and (2.2) (see the Appendix). As a result, we obtain from (2.6) the law of conservation of momentum of the nematic liquid crystal

$$\frac{\partial}{\partial t}(\rho\mathrm{v}_i)+\frac{\partial}{\partial x_j}[\rho\mathrm{v}_i\mathrm{v}_j-\sigma_{ij}^e]=0,\quad \sigma_{ij}^e=-\left(p\delta_{ij}+\frac{\partial(\rho F)}{\partial n_{k,j}}n_{k,i}\right) \tag{3.1}$$

Here σ^e_{ij} are the components of the Ericksen stress tensor [2, 5], $p = \rho^2(\partial F/\partial\rho)$ is the hydrodynamic pressure while $(\partial(\rho F)/\partial n_{k,j})n_{k,i}$ is the contribution of the microrotations of the particles in the stress field of the liquid crystal. Ignoring the last term in the Ericksen tensor, expression (3.1) is identical with the law of conservation of momentum of an ideal fluid.

After a similar procedure of changing to hydrodynamic variables, we obtain from (2.7) the following differential form of the equation of the balance of angular momentum

$$\frac{\partial}{\partial t}(J\rho\omega_i)+\frac{\partial}{\partial x_j}(J\rho\omega_i\omega_j)=\varepsilon_{ski}n_s h_k,\quad h_k=\frac{\partial\rho\mathbf{F}}{\partial n_k}-\frac{\partial}{\partial x_j}\left(\frac{\partial\rho\mathbf{F}}{\partial n_{k,j}}\right) \tag{3.2}$$

where h_k is the so-called molecular field, characterizing the presence of the momenta of the inertial forces in the nematic liquid crystal.

It should be noted that, on the left-hand side of (3.2) there is only the kinetic moment related to the rotation of the director n. The term related to the kinetic moment of an ideal fluid $\rho\mathbf{r}\times\mathbf{v}$ remains unchanged and does not occur in (3.2).

We can change from the laws of variation of the momentum and angular momentum to the equations of motion of the nematic liquid crystal in Euler variables

$$\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial \sigma^e_{ij}}{\partial x_j},\quad \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)=0$$

$$\rho J\left(\frac{\partial \omega_i}{\partial t}+v_j\frac{\partial \omega_i}{\partial x_j}\right)=\varepsilon_{ski}n_s h_k \tag{3.3}$$

closing them by the equation of state of the liquid (the Tait equation)

$$p=p_*[(\rho/\rho_0)^\Gamma-1] \tag{3.4}$$

where p_* is the internal pressure in the liquid, ρ_0 is the density of the unperturbed medium while Γ is a non-linear parameter. They are known in this form as the hydrodynamic equations of liquid crystals [2, 6].

4. INTERACTION BETWEEN THE HYDRODYNAMIC FIELDS AND THE WAVE DIRECTOR

As an example, we will derive the equations which describe the propagation of a one-dimensional torsional wave of orientation in a nematic liquid crystal, which is in a constant magnetic field $\mathbf{H}=\{0, H_0, 0\}$.

Suppose the director depends on one spatial coordinate $x_1=x$ and can rotate in the (x_2, x_3) plane, i.e. $\mathbf{n}(x,t)=\{0, n_2(x,t), n_3(x,t)\}$ (see Fig. 1), while the velocity field has the form $\mathbf{v}=(x,t), 0, 0\}$. If we introduce the angle $\theta(x,t)$ between the direction of the director n and the magnetic field H_0, we obtain $n_2=\cos\theta$, $n_3=\sin\theta$, and the angular velocity of the director has a single component, directed along the x axis $\omega(x,t)=d\theta/dt$. The volume density of the free energy (1.2) is

$$\rho F=\rho F_0(\rho)+\frac{K_2}{2}\left(n_3\frac{\partial\theta}{\partial x}\right)^2-\frac{\Delta\chi}{2}H_0^2\cos^2\theta$$

where $\rho F_0(\rho)=p_*(\Gamma-1)(\rho/\rho_0)^\Gamma+1]$ is the hydrodynamic part of the free energy. The non-zero components of the Ericksen stress tensor σ^e_{11} and the components of the molecular field $h_{2,3}$ have the form

$$\sigma^e_{11}=p+K_2\left(\frac{\partial\theta}{\partial x}\right)^2,\quad h_2=\Delta\chi H_0^2\cos\theta+K_2\frac{\partial^2}{\partial x^2}\cos\theta,\quad h_3=K_2\frac{\partial^2}{\partial x^2}\sin\theta$$

Substituting the expressions obtained into (3.3) and (3.4), we obtain the equations of motion of the nematic liquid crystal

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{K_2}{4\rho}\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)^2,\quad \frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \tag{4.1}$$

$$J\rho\frac{d^2\varphi}{dt^2}-K_2\frac{\partial^2\varphi}{\partial x^2}+\Delta\chi H_0^2\sin\varphi=0 \tag{4.2}$$

where we have put $\varphi=2\theta$. Equation (4.2) describes a torsional wave of the director of the nematic liquid crystal [7] and, when the medium is incompressible ($\rho=\rho_0$), is identical to the well-known sine-Gordon equation. If the kinetic energy of the director is negligibly small compared with its "potential" energy (i.e. $J\rho(\partial\varphi/\partial t)^2 \ll \rho(\partial\varphi/\partial x)^2$, $v=0$), we can neglect the first term in (4.2), and it will describe the quasi-static process of orientation of the director in a magnetic field, which is known as a Fredericks transition [2].

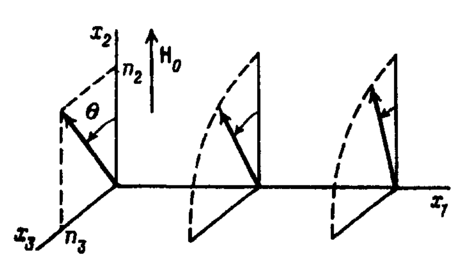

Fig. 1.

5. APPENDIX

The relation between variations in the Lagrange and Euler variables. The value of the variation, calculated for constant Euler coordinates

$$\delta_x f(x,t)=(\partial f(x,\ t,\ \varepsilon)/\partial\varepsilon)_{\varepsilon=0,x=\mathrm{const}}d\varepsilon \tag{5.1}$$

is related to the variation $\delta_\xi f(\mathbf{x},t)$, calculated for constant Lagrange coordinates ξ_α, as follows:

$$\delta_\xi f(x,t)=(\partial f[x_i(\xi_\alpha,\ t,\ \varepsilon),\ t,\ \varepsilon]/\partial\varepsilon))_{\varepsilon=0,\xi=\mathrm{const}}d\varepsilon=\delta_x f+\delta_\xi x_i\frac{\partial f}{\partial x_i} \tag{5.2}$$

where $\delta_\xi x_i$ is the Lagrangian of the variation of the Euler coordinate. The identity $\delta_\xi\xi_\alpha(x_i,t)=\delta_x\xi_\alpha+\delta_\xi x_i\partial\xi\alpha/\partial x_i\equiv 0$ and the equality $\delta_\xi x_i=-\partial x_i/\partial\xi_\alpha\delta_x\xi_\alpha$ follow from definition (5.2).

The Euler variation δ_x permutes with the partial derivatives in Euler variables $\partial/\partial x_i$ and $\partial/\partial t$, but does not commute with the derivatives $\partial/\partial\xi_\alpha$ and d/dt in the Lagrange system of coordinates. Formulae relating the Euler variations of the derivatives $\delta_x(d/dt)$ and $\delta_x(\partial/\partial\xi_\alpha)$ to the operators $\delta_x(d/dt)$ and $\delta_x(\partial/\partial\xi_\alpha)$ follow from definition (5.2) and, taking the above identity and equality into account, can be written in the form

$$\left(\delta_x\frac{d}{dt}\right)f=\left(\frac{d}{dt}\delta_x\right)f-\frac{d}{dt}\left(x_{i,\alpha}\frac{\partial f}{\partial x_i}\delta_x\xi_\alpha\right)+\frac{\partial}{\partial x_i}\left(\frac{df}{dt}\right)x_{i,\alpha}\delta_x\xi_\alpha \tag{5.3}$$

$$\left(\delta_x\frac{\partial}{\partial\xi_\alpha}\right)f=\left(\frac{\partial}{\partial\xi_\alpha}\delta_x\right)f-\frac{\partial}{\partial\xi_\alpha}\left(x_{i,\beta}\frac{\partial f}{\partial x_i}\delta_x\xi_\beta\right)+\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial\xi_\alpha}\right)x_{i,\beta}\delta_x\xi_\beta \tag{5.4}$$

The momentum transfer equation. We now consider Eq. (2.6) with components of the non-degenerate matrix $\xi_{\alpha,i}=(\partial\xi_\alpha/\partial x_i)$ and further add and subtract the following quantity from the expression obtained

$$\frac{\partial L}{\partial n_k}n_{k,i}+\frac{\partial L}{\partial n_{k,j}}\frac{\partial}{\partial x_i}(n_{k,j})+\frac{\partial L}{\partial\dot n_k}\frac{\partial}{\partial x_i}(\dot n_k)$$

After grouping terms we have

$$\begin{aligned}&\frac{\partial L}{\partial\xi_{\alpha,j}}\frac{\partial}{\partial x_i}(\xi_{\alpha,j})+\frac{\partial L}{\partial\xi_{\alpha,t}}\frac{\partial}{\partial x_i}(\xi_{\alpha,t})+\frac{\partial L}{\partial n_k}\frac{\partial}{\partial x_i}(n_k)+\frac{\partial L}{\partial n_{k,j}}\frac{\partial}{\partial x_i}(n_{k,j})+\\&+\frac{\partial L}{\partial\dot n_k}\frac{\partial}{\partial x_i}(\dot n_k)-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial\xi_{\alpha,j}}\xi_{\alpha,i}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\xi_{\alpha,t}}\xi_{\alpha,i}\right)-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial n_{k,j}}n_{k,i}\right)-\\&-\left[\frac{\partial L}{\partial n_k}-\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial n_{k,j}}\right)-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot n_k}\right)-\frac{\partial L}{\partial\dot n_k}\mathrm{v}_{m,m}\right]n_{k,i}=0\end{aligned} \tag{5.5}$$

The first six terms form a partial derivative with respect to the coordinate x_i of the Lagrangian $L=L[\xi_{\alpha,i}(x_i,t),\ \xi_{\alpha,t},n_k(x_i,t),\dot n_k,n_{k,j}]$, while the expression in square brackets, by (2.7), is identically equal to zero, and hence from (5.5) we obtain the divergent form of the conservation law

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\xi_{\alpha,t}}\xi_{\alpha,t}\right)+\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial\xi_{\alpha,j}}\xi_{\alpha,i}+\frac{\partial L}{\partial n_{k,j}}n_{k,i}-L\delta_{ij}\right)=0 \tag{5.6}$$

Using the expression for the Lagrangian (1.3) and also relations (2.1) and (2.2), which relate the velocity and density fields to the derivatives of the Lagrangian variables, we obtain that

$$\frac{\partial L}{\partial\xi_{\alpha,t}}\xi_{\alpha,i}=-\rho\mathrm{v}_i$$

$$\frac{\partial L}{\partial\xi_{\alpha,j}}\xi_{\alpha,i}+\frac{\partial L}{\partial n_{k,j}}n_{k,i}-L\delta_{ij}=-\rho\mathrm{v}_i\mathrm{v}_j-\rho^2\frac{\partial F}{\partial\rho}\delta_{ij}-\frac{\partial(\rho F)}{\partial n_{k,j}}n_{k,i}$$

substituting these inequalities into (5.6) we obtain the law of conservation of momentum (3.1).

The kinetic moment balance equation. We now consider Eq. (2.7) with non-degenerate matrix $\varepsilon_{ski}n_s$ and further using relations (2.1), (2.2) and (1.3) we change to Euler variables in the following expressions

$$\frac{\partial L}{\partial n_k}\in_{ski}n_s=\frac{\partial(\rho F)}{\partial n_k}\in_{ski}n_s,\quad \frac{\partial L}{\partial \dot{n}_k}\mathrm{v}_{m,m}\in_{ski}n_s=\rho J\omega_i\mathrm{v}_{m,m}$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial L}{\partial n_{k,j}}\right)\varepsilon_{ski}n_s=-\frac{\partial}{\partial x_j}\left(\frac{\partial(\rho F)}{\partial n_{k,j}}\right)\varepsilon_{ski}n_s$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{n}_k}\right)\in_{ski}n_s=\rho J\dot{\omega}_i+\dot{\rho}J\omega_i$$

Here we have taken into account the fact that $\varepsilon_{ijk}\varepsilon_{lmk}=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})$ and $\dot{\omega}_i=\varepsilon_{ski}n_s\ddot{n}_k$. After this the expression obtained is the balance equation (3.2).

We wish to thank E. L. Aero for data on the hydromechanics of liquid crystals and G. A. Maugin for drawing our attention to the theory of oriented elastic bodies and for kindly providing a reprint of the paper [8].

This research was partially supported financially by the International Science Foundation and the Russian Government (R9B300), the Russian Foundation for Basic Research (95-02-05360) and the International Association for Promoting Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS-96-2370).

REFERENCES

1. KATS, Ye. I. and LEBEDEV, V. V., *The Dynamics of Liquid Crystals*. Nauka, Moscow, 1988.
2. SONIN, A. S., *Introduction to Liquid Crystal Physics*. Nauka, Moscow, 1953.
3. SELIGER, R. L. and WHITHAM, G. B., Variational principles in continuum mechanics. *Proc. Roy. Soy. Ser. A*, 1968, **305**, 1480, 1–25.
4. BERDICHEVSKII, V. L., *Variational Principles in Continuum Mechanics*. Nauka, Moscow, 1983.
5. LISIN, V. B. and POTAPOV, A. I., Variational principle in the mechanics of liquid crystals. *Int. J. Non-linear Mech*. 1997, **32**, 1, 55–62.
6. AERO, E. L. and BULYGIN, A. N., The hydromechanics of liquid crystals. In *Advances in Science and Technology. Ser. Hydrodynamics*. Vsesoyuz. Inst. Nauch. i Tekh. Inform., 1973, **7**, 106–213.
7. KAMENSKII, V. G., The non-linear dynamics of the director of nematics in a magnetic field. *Zh. Eksp. Teor. Fi*., 1984, **87**, 4, 1262–1276.
8. POUGET, J., MAUGIN, G. A., Nonlinear dynamics of oriented elastic solids. I. Basic equations. *J. Elasticity*. 1989, **22**, 2/3, 135–155.

Translated by R.C.G.